MTH 264 Introduction to Matrix Algebra - Summer 2023.

LN3C. About Reflection and Projection Transformations and Determinants.

These lecture notes are mostly lifted from the text **Matrix and Power Series, Lee and Scarborough, custom 5th edition**. This document highlights parts of the text that are used in the lecture sessions.

Part 1. Moar about Reflections and Orthogonal Projections on $\mathbb{R}^3$

We have discussed using the change-of-basis method to find left multiplication matrices for reflections and orthogonal projections on $\mathbb{R}^3$. However, finding inverses is generally tedious to do by hand. We present another way of finding these left multiplication matrices.

Theorem 3C.1. Reflections on $\mathbb{R}^3$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a reflection across the plane P with normal vector $\mathbf{N} \in \mathbb{R}^3$ passing through the origin. Then, its left multiplication matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{I_3} - 2\mathbf{n}\mathbf{n}^\top \text{ for } \mathbf{n} = \frac{1}{||\mathbf{N}||}\mathbf{N} \quad \text{or more generally} \quad \mathbf{A} = \frac{1}{\langle \mathbf{N}, \mathbf{N} \rangle}\Big(\langle \mathbf{N}, \mathbf{N} \rangle \mathbf{I_3} - 2\mathbf{N}\mathbf{N}^\top\Big)$$

In the literature, this is called a Householder reflection. This formula also generalizes to hyper-planes on higher dimensions with normal vector $\mathbf{N}$.

This is a consequence of the dot product being equivalent to a specific matrix operation, i.e.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{x}$$

with vectors presented as column vectors. Then, a reflection transformation T across the plane with normal $\mathbf{N}$ satisfies

$$\begin{aligned} T(\mathbf{x}) &= \mathbf{x} - 2\,\mathrm{proj}_{\mathbf{N}}(\mathbf{x}) = \mathbf{x} - 2\frac{\langle \mathbf{x}, \mathbf{N} \rangle}{\langle \mathbf{N}, \mathbf{N} \rangle}\mathbf{N} = \mathbf{x} - \frac{2}{\langle \mathbf{N}, \mathbf{N} \rangle}(\mathbf{N}^\top \mathbf{x})\mathbf{N} = \mathbf{I_3}\mathbf{x} - \frac{2}{\langle \mathbf{N}, \mathbf{N} \rangle}\mathbf{N}\mathbf{N}^\top \mathbf{x} \\ &= \frac{1}{\langle \mathbf{N}, \mathbf{N} \rangle}\Big(\mathbf{I_3} - 2\mathbf{N}\mathbf{N}^\top\Big)\mathbf{x} = \mathbf{A}\mathbf{x} \qquad \text{with} \qquad \mathbf{A} = \frac{1}{\langle \mathbf{N}, \mathbf{N} \rangle}\Big(\mathbf{I_3} - 2\mathbf{N}\mathbf{N}^\top\Big) \end{aligned}$$

We can adapt this method to determine left multiplication matrices for orthogonal projections too.

Theorem 3C.2. Projections on $\mathbb{R}^3$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal projection along the plane P with normal vector $\mathbf{N} \in \mathbb{R}^3$ passing through the origin. Then, its left multiplication matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{I_3} - \mathbf{n}\mathbf{n}^\top \text{ for } \mathbf{n} = \frac{1}{||\mathbf{N}||}\mathbf{N} \quad \text{or more generally} \quad \mathbf{A} = \frac{1}{\langle \mathbf{N}, \mathbf{N} \rangle}\Big(\langle \mathbf{N}, \mathbf{N} \rangle \mathbf{I_3} - \mathbf{N}\mathbf{N}^\top\Big)$$

This formula generalizes to hyper-planes on higher dimensions with normal vector $\mathbf{N}$.

Here, the transformation satisfies $T(\mathbf{x}) = \mathbf{x} - \mathrm{proj}_{\mathbf{N}}(\mathbf{x})$, eliminating the component of $\mathbf{x}$ along the normal.

Part 2. Moar Determinants

Some definitions for special matrices.

Definition 3C.3. Triangular Matrices
Let $\mathbf{A} \in \mathbb{R}^n$ be a square matrix. Then,

- **(a)** We say that $\mathbf{A}$ is **upper triangular** if all entries below the main diagonal are zero.
- **(b)** We say that $\mathbf{A}$ is **lower triangular** if all entries above the main diagonal are zero.
- **(c)** We say that $\mathbf{A}$ is **triangular** if $\mathbf{A}$ is either upper triangular or lower triangular.

For this course, we primarily use the determinant in relation to linear systems and linear independence. We state the result below.

Theorem 3C.4. Characterizations involving Determinants
Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a matrix and let $V = \{\mathbf{a_1}, \dots, \mathbf{a_n}\}$ of vectors in $\mathbb{R}^n$ with $\mathbf{a_i}$ corresponding to the i^{th} column of $\mathbf{A}$. The following are equivalent:

- **(a)** $\det(\mathbf{A}) \neq 0$.
- **(b)** The matrix $\mathbf{A}$ is invertible and $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$.
- **(c)** V is linearly independent.
- **(d)** V is a basis for $\mathbb{R}^n$. i.e. V is linearly independent and $\text{span}(V) = \mathbb{R}^n$.
- **(e)** The system $\mathbf{Ax} = \mathbf{b}$ for any vector $\mathbf{b} \in \mathbb{R}^n$ has exactly one solution.
- **(f)** The row echelon form of $\mathbf{A}$ has a pivot for each column.

Conversely, we also have the following equivalence:

- **(a)** $\det(\mathbf{A}) = 0$.
- **(b)** The matrix $\mathbf{A}$ is not invertible.
- **(c)** V is linearly dependent.
- **(d)** $\text{span}\, V \neq \mathbb{R}^n$, i.e. there are vectors $x \in \mathbb{R}^n$ that cannot be represented as a linear combination of V.
- **(e)** For all $\mathbf{b} \in \mathbb{R}^n$, the system $\mathbf{Ax} = \mathbf{b}$ is either (1) inconsistent, i.e. no solutions, or (2) consistent with infinitely many solutions.
- **(f)** There exists a vector $\mathbf{b} \in \mathbb{R}^n$ such that the system $\mathbf{Ax} = \mathbf{b}$ has no solution.
- **(g)** There exists a vector $\mathbf{b} \in \mathbb{R}^n$ such that the system $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions.
- **(h)** The row echelon form of $\mathbf{A}$ has at least one column with no pivot.

Theorem 3C.5. Properties of Determinants
Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix.

1. **Identity Matrices have determinant 1**: $\det(\mathbf{I_n}) = 1$.
2. **Homogeneity of Degree n**: For all scalars $k \in \mathbb{R}$, $\det(k\mathbf{A}) = k^n \det(\mathbf{A})$.
3. **Multilinearity**: $\det(\mathbf{B}) = k \det(\mathbf{A})$ where $\mathbf{B}$ is the result of replacing the i^{th} row $\mathbf{R_i}$ of $\mathbf{A}$ a scalar

multiple $k\mathbf{R_i}$ with $k \in \mathbb{R}^n$.

4. **Distributivity over Matrix Mult.**: For any matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$, $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{BA})$.
5. **Invariance under Transposition**: $\det(\mathbf{A}) = \det(\mathbf{A}^\top)$.
6. **Determinants of Triangular Matrices**: Assume $\mathbf{A}$ is a triangular matrix and let $a_{i,i}$ be the i^{th} entry of the main diagonal of $\mathbf{A}$, i.e. $a_{i,i}$ is the entry of $\mathbf{A}$ in row i and column i. Then, $\det A = a_{1,1}a_{2,2}\cdots a_{n,n}$. That is, the determinant of a triangular matrix is the product of its entries on the main diagonal.

We re-state some of these properties in terms of row operations.

Theorem 3C.6. Efficient Calculation of Determinants
Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Applying row operations on $\mathbf{A}$ affects $\det A$ in the following ways:

Type (I) Row Operations:	$\mathbf{A} \xrightarrow{\mathbf{R_i}\leftrightarrow\mathbf{R_j}} \mathbf{B}$ with $i \neq j$	implies	$\det(\mathbf{A}) = (-1)\det(\mathbf{B})$
Type (II) Row Operations:	$\mathbf{A} \xrightarrow{k\mathbf{R_i}\mapsto\mathbf{R_i}} \mathbf{B}$ with $k \neq 0$	implies	$\det(\mathbf{A}) = \dfrac{1}{k}\det(\mathbf{B})$
Type (III) Row Operations:	$\mathbf{A} \xrightarrow{\mathbf{x}+\mathbf{R_i}\mapsto\mathbf{R_i}} \mathbf{B}$ with $\mathbf{x} = k_1\mathbf{R_1} + \cdots + k_i\mathbf{R_i} + \cdots + k_n\mathbf{R_n}$ and $k_i = 1$	implies	$\det(\mathbf{A}) = \det(\mathbf{B})$

When doing calculation, it's easier to keep track of the multiplier m upon application of row operations.